**December 19th, 2020** **Time:10:00-1:00**

**Basic Probability Final Exam** **Email: sivaathreya@yahoo.co.in**

**Maximum Score : 100**

1. (10 points) A certain firm produces resistors and markets them as 10–ohm resistors. However the actual ohms of resistance produced by the resistors may vary. Research has established that 10% of the values are below 9.5 ohms and 20% are above 10.5 ohms. If two resistors, randomly selected, are used in a system, find the probabilities of the following events:

(a) Both resistors have actual values between 9.5 and 10.5 ohms.

(b) At least one resistor has an actual value in excess of 10.5 ohms.

2. (10 points) Customers arrive at a service counter. The waiting time or time taken to service is given by an exponential(1) random variable. Let 100 customers come to the service counter in a day and $S_{100}$ be the number of customers that wait for more than 10 minutes.

(i) Find the probability distribution of $S_{100}$.

(ii) Suppose we wish to approximate the probability that at least half of the customers that arrive in a day must wait for more than 10 minutes by P( $a < Z < b$) then identify the random variable $Z$, the numbers $-\infty \leq a \leq \infty$ and $-\infty \leq b \leq \infty$.

3.(10 points) A random variable $X$ has a density function of the form

$$f_X(x) = \begin{cases} cx^2, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

for some constant $c$. Find $c$ and $Var(X)$.

4. (20 points) Solve the following questions giving reasons. Please use complete sentences or precise mathematical statements while justifying your answer.

(a) Beginning at 12:00 midnight, a computer center is up for one hour and down for 2 hours on a regular cycle. A person who does not know the schedule dials the centre at a random time uniformly selected between 12:00 midnight and 5:00am. What is the probability that the centre will be operating when this call comes in ?

(b) Two random variables $X$ and $Y$ have the following joint probability distribution function:

| p(x,y) | $X = -1$ | $X = 0$ | $X = 2$ |
|---|---|---|---|
| $Y = 0$ | 0.1 | 0.2 | 0.1 |
| $Y = 1$ | 0 | 0.2 | 0 |
| $Y = 2$ | 0 | 0.1 | 0.3 |

Find Covariance$(X, Y)$.

(c) Which of the following functions is the graph of a cummulative distribution function ?

(d) Let $X$ be a Uniform(0,1) random variable. Let $Y \bigm| \{X = x\}$ be a Uniform$(-2x, x)$ random variable. Find the joint probability density function of $X$ and $Y$.

5. (20 points ) Each morning at 9 a.m., Siva on a desert island throws a random number of bottles containing pleas for rescue into the sea. Suppose that the number of bottles he throws in each morning has a Poisson distribution with parameter $\mu$ and is independent of what has gone on in the past. It is possible that any one of the bottles currently adrift (not just those thrown in that morning) will sink. The probability of any given bottle sinking in any given 24 hour period is $p$. Also, the behaviour of the individual bottles is independent. Let $X_i$ be the number of bottles still afloat on the Siva's $i^{\text{th}}$ morning on the island just before he throws in a new batch. In particular, $X_1 = 0$. Find the distribution of $X_n$.

6. (20 points) Suppose the random variables $X$ and $Y$ have a joint probability density function given by

$$f(x, y) = 3y$$

for all $(x, y)$ in the triangle bounded by $y = 0, y = 1 - x$ and $y = 1 + x$, and $f(x, y) = 0$ for all $(x, y)$ outside this region.

(a) Sketch the region where the density is non-zero.

(b) Find the probability density function of $X$.

(c) Find the probability density function of $U = \mid X \mid$.

7. (20 points) Let $T$ be an exponential random variable with mean $\lambda$. Find the probability density function of $Y = \frac{T}{1+T}$.